JOURNAL OF ALGEBRA 103, 377–380 (1986)

Gersten's Conjecture for the K-Theory with Torsion Coefficients of a Discrete Valuation Ring

HENRI GILLET*

Department of Mathematics, Princeton University, Princeton, New Jersey 08540

Communicated by Richard G. Swan

Received October 10, 1984

In [2] Gersten conjectured that if R is a regular local ring and $\mathbf{M}^{(i)}(R)$ is the abelian category of finitely generated R modules with support having codimension at least i in $\operatorname{Spec}(R)$, the homomorphisms, for $q \geqslant 0$,

$$K_q(\mathbf{M}^{(i+1)}(R)) \to K_q(\mathbf{M}^{(i)}(R))$$

between Quillen K-groups induced by the obvious inclusion functors, are all zero. In [6] Quillen proved this conjecture if R is essentially of finite type over a field. In the case of discrete valuation rings, this conjecture reduces to:

Conjecture. Let R be a discrete valuation ring with residue field k and field of fractions F. Then for all $q \geqslant 0$, the transfer map (which is defined since every k vector space is of finite projective dimension as an R module):

$$K_q(k) \to K_q(R)$$

is zero. Equivalently, the map

$$K_q(R) \to K_q(F)$$

induced by the ring homomorphism $R \to F$ is injective. A third equivalent formulation is that the boundary map:

$$K_{q+1}(F) \xrightarrow{\partial} K_q(k)$$

is surjective for all $q \geqslant 0$. That the three formulations are equivalent is a consequence of the localization sequence:

$$\cdots \to K_{q+1}(F) \xrightarrow{\partial} K_q(k) \to K_q(R) \to K_q(F) \to \cdots.$$

* Current address: University of Illinois, Chicago, Illinois 60680.

0021-8693/86 $3.00

In the non-equicharacteric case, this conjecture has been proved by Gersten if k is finite [2] and by C. Sherman if k is the algebraic closure of a finite field [9]; see also [7] for the equicharacteristic case. In [4] the author and M. Levine proved that the general conjecture for local rings smooth over a discrete valuation ring could be reduced to the case of a discrete valuation ring.

In this short note we shall prove the analogous statement for K-theory with coefficients (for the definition of $K(\ ,Z/n)$ see [5, 10]).

THEOREM A. *Let R be a discrete valuation ring with residue field k. Let $n \in N$ with* char(k) *not dividing n; then the transfer map*

$$K_q(k, Z/n) \to K_q(R, Z/n)$$

is the zero map, for all $q \geqslant 0$.

By the methods of [4] one immediately obtains:

COROLLARY B. *If R is a regular local ring, smooth over a discrete valuation ring, then the homomorphisms:*

$$K_q(\mathbf{M}^{(i)}(R), Z/n) \to K_q(\mathbf{M}^{(i-1)}(R), Z/n)$$

are all zero.

The theorem is a consequence of a theorem of Suslin, together with the following technical lemma:

PROPOSITION C. *In order to prove Gersten's conjecture for a discrete valuation ring R with residue field k, it is enough to show that for any $\alpha \in K_q(k, \Lambda)$ ($\Lambda = Z$ or Z/n), there exists an extension of discrete valuation rings with residue field k:*

$$\begin{tikzcd} R' \arrow[dr, "\varepsilon'"] & \\ \bigcup & \\ R \arrow[r, two heads, "\varepsilon"] & k \end{tikzcd}$$

such that R'/R is quasi-finite (i.e., the fraction field F' of R' is a finite extension of the fraction field F of R) and of finite type, for which α "lifts over R'," i.e., there exists $\tilde{\alpha} \in K_q(R', \Lambda)$ with $\varepsilon'_*(\tilde{\alpha}) = \alpha$.

Proof of Proposition C. Let A/R be a finite extension such that R' is a localization of A at a single maximal ideal $\mathfrak{p}$. If F'/F is separable we may take A to be the integral closure of R in F', while in general A exists by Zariski's Main Theorem; note that A is semi-local of dimension one, with fraction field F'. Let $\mathfrak{p}=\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the maximal ideals of A. We have a map of localization sequences induced by the exact functors $\mathbf{M}^{(i)}(A) \to \mathbf{M}^{(i)}(R)$ for $i \geqslant 0$, coming from restriction of scalars (cf. [3, p. 275]):

$$\begin{array}{ccccccccc}
\to K_{q+1}(F', \Lambda) & \xrightarrow{\partial'} & \bigoplus_{i=0}^{r} K_q(A/\mathfrak{p}_i, \Lambda) & \to & K'_q(A, \Lambda) & \to & K_q(F', \Lambda) & \to \\
\downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle \tau} & & \downarrow & & \| & \\
\to K_{q+1}(F, \Lambda) & \xrightarrow{\partial} & K_q(k, \Lambda) & \to & K_q(R, \Lambda) & \to & K_q(F, \Lambda) & \to
\end{array}.$$

We want to prove that α is in the image of ∂. Since $K_q(A/\mathfrak{p} \xrightarrow{\sim} k, \Lambda) \to K_q(k, \Lambda)$ is the identity map, it is enough to exhibit a class $\beta \in K_{q+1}(F', \Lambda)$ such that

$$\partial'(\beta) = \alpha \oplus 0 \in K_q(A/\mathfrak{p}, \Lambda) \oplus \bigoplus_{i=1}^{k} K_q(A/\mathfrak{p}_i, \Lambda).$$

By the Chinese Remainder Theorem, we may choose an $f \in \mathfrak{p}$ which generates the ideal $\mathfrak{p}A_{\mathfrak{p}}$ (recall that $\mathfrak{p}A_{\mathfrak{p}}$ is the maximal ideal of the discrete valuation ring R') such that $f \equiv 1(\mathfrak{p}_i)$ for $i=1, \ldots, k$. There is a product:

$$K_p(F', \Lambda) \otimes K_q(F') \to K_{p+q}(F', \Lambda)$$

and we set $\beta = u_*(\tilde{\alpha}) * \{f\} \in K_{q+1}(F', \Lambda)$, where $u\colon R' \to F'$ is the inclusion. We may compute $\partial'(\beta)$ as a sum:

$$\partial_{\mathfrak{p}}(\beta) \oplus \sum_{i=1}^{k} \partial_{\mathfrak{p}_i}(\beta)$$

where each $\partial_{\mathfrak{p}_i}$ is the boundary map for the localization sequence:

$$K'_{q+1}(A_{\mathfrak{p}_i}, \Lambda) \to K_{q+1}(F', \Lambda) \xrightarrow{\partial_{\mathfrak{p}_i}} K_q(A/\mathfrak{p}_i, \Lambda) \to.$$

The computation is in two parts:

For $i=0$, $\partial_{\mathfrak{p}}(u_*(\tilde{\alpha}) * \{f\}) = (-1)^{q+1} a * \partial_{\mathfrak{p}}(f) = (-1)^{q+1}\alpha$ by [3, 7.14] since $\tilde{\alpha} \in A_{\mathfrak{p}} = R'$ and f generates $\mathfrak{p}A_{\mathfrak{p}}$.

For $i > 0$,

$$\begin{aligned}
\partial_{\mathfrak{p}_i}(u_*(\tilde{\alpha}) * f) &= \partial_{\mathfrak{p}_i}(u_*(\tilde{\alpha})) * (\{f\}(\mathfrak{p}_i)) \\
&= \partial_{\mathfrak{p}_i}(u_*(\tilde{\alpha})) * \{1\} \\
&= 0.
\end{aligned}$$

Hence

$$\partial(N_{F'/F}(\beta)) = \tau(\partial'(\beta)) = (-1)^{q+1}\alpha,$$

and the proof is complete.

Proof of the Theorem. By a result of A. Suslin [11] we know that if R is Henselian, and char(k) does not divide n, then

$$K_q(R, Z/n) \simeq K_q(k, Z/n).$$

Now suppose that R is a discrete valuation ring with residue field k; let $\tilde{R}$ be the Henselization of R. Then $K_q(\tilde{R}, Z/n) = \varinjlim_{R'} K_q(R', Z/n)$, where the direct limit is over all R'/R discrete valuation rings etale over R, and we can apply Proposition C.

Remark. From proposition C, in order to prove Gersten's conjecture for dvr's, and hence by [4] for all local rings smooth over a dvr, it would be enough to show that if X is a finite CW complex then any representation

$$\pi_1(X) \to GL_n(k)$$

can be lifted, as a virtual representation, to some discrete valuation ring R'/R as in the proposition. This follows by an argument as in [8] or [1, Theorem 2.5]. In particular, as C. Sherman points out in [10], if k is finite such a lifting always exists, leading to a simpler proof than that of [2] of the conjecture in that case.

References

1. S. M. Gersten, Higher K-theory of rings, *in* "Algebraic K-Theory I" (H. Bass, Ed.), Lecture Notes in Mathematics Vol. 341, pp. 3–42, Springer-Verlag, Berlin.
2. S. M. Gersten, Some exact sequences in the higher K-theory of rings, *in* "Algebraic K-Theory I" (H. Bass, Ed.), Lecture Notes in Mathematics Vol. 341, pp. 211–243, Springer-Verlag, Berlin.
3. H. Gillet, Riemann Roch theorems for higher algebraic K-theory, *Advances in Math.* **40** (1981), 203–289.
4. H. Gillet and M. Levine, The relative form of Gersten's conjecture over a discrete valuation ring: the smooth case, *J. Pure Appl. Algebra*, to appear.
5. J. Niesendorfer, Primary homotopy theory, *Mem. Amer. Math. Soc.* **232** (1980).
6. D. Quillen, Higher algebraic K-theory I, *in* "Algebraic K-theory, I" (H. Bass, Ed.), Lecture Notes in Mathematics Vol. 341, pp. 169–198, Springer-Verlag, Berlin, 1973.
7. C. Sherman, The K-theory of an equicharacteristic discrete valuation ring injects into the K-theory of its field of quotients, *Pacific J. Math.* **74** (1978), 497–499.
8. C. Sherman, Some splitting results in the K-theory of rings, *Amer. J. Math.* **101** (1979), 609–632.
9. C. Sherman, Group representations and algebraic K-theory, *in* "Algebraic K-Theory, Oberwolfach, 1980," Lecture Notes in Mathematics Vol. 966, Springer-Verlag, New York, 1982.
10. C. Sherman, Some splitting results in the K-theory of rings, II, preprint, 1985.
11. A. A. Suslin, On the K-theory of local fields, *J. Pure Appl. Algebra* **34** (1984), 301–318.